International Journal of Theoretical Physics, Vol. 6, No. 1 (1972), pp. 37–45

Field Theory Where All Γ^i_{jk} Invariants Vanish

M. MURASKIN

University of North Dakota, Grand Forks, North Dakota

Received: 24 *July* 1971

Abstract

We investigate whether a nontrivial field theory exists for which all scalars formed from Γ^i_{jk} (where g_{ij} is used to raise and lower indices) can be zero for all points of space and time. We find some examples for which the invariants are all zero for all points at which the field is finite. We also comment upon the problem of boundary conditions, in general.

1. *Introduction*

In our previous studies (Muraskin, 1970; 1971a and b; Muraskin & Clark, 1970; Muraskin & Ring, 1971, 1972; Muraskin, 1972) of $\Gamma^i_{jk;l} = 0$, $g_{ij;k} = 0$ we did not concern ourselves with the problem of boundary conditions. This is a reasonable working hypothesis if:

(a) Boundary conditions automatically take care of themselves once we make a reasonable choice for our parameters at the initial point. We recall that the field is determined uniquely once the parameters at the initial point are specified.
(b) Turnabout of field components around the point where the component is a maximum (minimum) is not very sensitive to behavior at infinity.

In our last paper, we found indications that these favorable possibilities above are somewhat unlikely. That is, the computer solutions show a tendency toward blow-up as we go further from the origin point. It is still not impossible that this situation may reverse itself as we proceed still farther from the origin. However, at this point we have no reason to believe in this latter possibility. Thus, it may well be that the boundary condition problem cannot safely be ignored.

A natural set of boundary conditions is that all invariants involving Γ^i_{jk} should be zero at the origin point. We have already remarked that all invariants constructed from Γ^i_{jk}, g_{ij} and ∂_k are constants as a consequence of the field equations. This means that our boundary conditions imply that all invariants involving Γ^i_{jk} are zero everywhere, including at infinity.

Now, there are an infinite number of invariants involving Γ^i_{jk}. Thus, it is far from obvious that there exists an initial set of data (Γ^i_{jk}, g_{ij}) such that all infinite invariants are zero. Furthermore, to make matters even more difficult, the initial data must satisfy ninety-six algebraic non-linear integrability conditions. Thus, it is not clear whether we can construct a theory in which these conditions can all be met.

In this paper, we shall prove that a set of Γ^i_{jk}, g_{ij} do, in fact, exist at the origin point such that all invariants constructed from them (involving Γ^i_{jk}) are zero at all points (for which the field is finite). In addition, the integrability equations are satisfied.

2. *First Example*

We write, as in our previous papers,

$$\Gamma^i_{jk} = e_\alpha{}^i e^\beta{}_j e^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \tag{2.1}$$

$$g_{ij} = e^\alpha{}_i e^\beta{}_j g_{\alpha\beta} \tag{2.2}$$

where $g_{\alpha\beta}$ is the Minkowski metric. An invariant formed from $\Gamma^\alpha_{\beta\gamma}, g_{\alpha\beta}$ will be unchanged by an $e^\alpha{}_i$ transformation. For example,

$$g_{ij} \Gamma^j_{km} \Gamma^i_{np} g^{kp} g^{mn} = g_{\alpha\beta} \Gamma^\beta_{\rho\delta} \Gamma^\alpha_{\chi\lambda} g^{\rho\lambda} g^{\delta\chi} \tag{2.3}$$

Thus, we can work with the simpler set $\Gamma^\alpha_{\beta\gamma}$, $g_{\alpha\beta}$, and then transform to get g_{ij}, Γ^i_{jk}. The invariants are unchanged by this procedure.

Our choice of $\Gamma^\alpha_{\beta\gamma}$ is

$$\begin{aligned}
\Gamma^1_{11} &= \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^0_{10} = a \\
\Gamma^1_{21} &= \Gamma^2_{22} = \Gamma^3_{23} = \Gamma^0_{20} = b \\
\Gamma^1_{31} &= \Gamma^2_{32} = \Gamma^3_{33} = \Gamma^0_{30} = c \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \Gamma^0_{00} = d
\end{aligned} \tag{2.4}$$

This set satisfies the integrability equations $R^i_{jkl} = 0$ for any choice of a, b, c, d. We shall show that the set (2.4) implies that all invariants involving Γ^i_{jk} are zero if the following relation is met

$$a^2 + b^2 + c^2 - d^2 = 0 \tag{2.5}$$

We shall change our notation for the sake of simplicity. For example, we shall write

$$g_{\alpha\beta} \Gamma^\alpha_{\rho\chi} \Gamma^\beta_{\lambda\sigma} g^{\chi\lambda} g^{\rho\sigma} \tag{2.6}$$

as

$$\Gamma_{\alpha\beta\gamma} \Gamma_{\alpha\gamma\beta} \tag{2.7}$$

This notation is used often when one works in a Minkowski space. This is the case, here, since $g_{\alpha\beta}$ is the Minkowski metric. We have to remember that when the summation involves the zero component, we get a relative sign appearing compared to the case that the indices are 1, 2, 3.

From the form (2.4), we see the first index is always the same as the third. Thus (2.7) becomes

$$\Gamma_{\alpha\alpha\alpha}\Gamma_{\alpha\alpha\alpha} \tag{2.8}$$

Now, $\Gamma_{111} = -a$, $\Gamma_{222} = -b$, $\Gamma_{333} = -c$, $\Gamma_{000} = d$. Thus (2.8) gives $-a^2 - b^2 - c^2 + d^2$ which is to be zero (see (2.5)). We can check from (2.6) that the signs are correct.

In a general product of gammas, we can see what types of terms appear. We denote by α an index that appears in the middle position of one of the gammas. A possible kind of terms is

$$\Gamma_{\sigma\alpha\sigma}\Gamma_{\tau\alpha\tau}f(\tau, \sigma, \ldots) \tag{a}$$

$f(\tau, \sigma, \ldots)$ involves products of gammas involving τ, σ as well as other indices. We evaluate the summation over α keeping first σ, τ, ... fixed. Now, from (2.4) the value of $\Gamma_{\sigma\alpha\sigma}$† is the same independent of which σ we choose. Thus, we get

$$(-a^2 - b^2 - c^2 + d^2)f(\tau_1, \sigma_1, \ldots)$$

where τ_1 and σ_1 denotes the particular choices of τ and σ. This expression is zero for all τ and all σ. Thus, the combination (a) appearing in a product of gammas is always zero.

We point out that combinations like

$$\Gamma_{\sigma\alpha\sigma}\Gamma_{\tau\alpha\tau}\Gamma_{\nu\alpha\nu}\Gamma_{\rho\alpha\rho}$$

cannot appear in the product of gammas. That is, if we have two α's appearing in the middle positions, then no other middle place can have the α index. This result can be traced back to the statement that the equality between indices is for the first index and the third. We shall have more to say about this rule later on.

We see that the number of α's appearing must always be even.

Another possible kind of term is

$$\Gamma_{\sigma\alpha\sigma}\Gamma_{\tau\alpha\tau}(\Gamma_{\alpha\rho\alpha}\Gamma_{\alpha\beta\alpha}\Gamma_{\alpha\nu\alpha}\ldots)f(\sigma, \tau, \rho, \beta, \nu, \text{etc.}) \tag{b}$$

The dots refer to additional terms of the structure $\Gamma_{\alpha\delta\alpha}$. Again, we fix $\sigma, \tau, \rho, \beta, \nu$ and do the summation over α first. Since $\Gamma_{\alpha\rho\alpha}$ is the same independent of α, we get (for example, we take $\rho = 1$, $\beta = 2$, $\nu = 2$)

$$(-ab^2\ldots)(-a^2 - b^2 - c^2 + d^2)f(\sigma, \tau, \rho = 1, \beta = 2, \nu = 2, \text{etc.})$$

This expression is again zero for all values of σ, τ, ρ, ν, β. (Note, f does not involve any denominators.)

Another kind of term is

$$\Gamma_{\alpha\alpha\alpha}\Gamma_{\alpha\alpha\alpha}f \tag{c}$$

We have already pointed out that this is zero.

† No summation over σ here. In similar contexts, later on, no summation will be implied. Of course, in all invariants, summation is implied.

Another type is

$$\Gamma_{\alpha\alpha\alpha}\Gamma_{\rho\alpha\rho}f(\rho, \ldots) \tag{d}$$

This gives $f(\rho)(-a^2 - b^2 - c^2 + d^2)$, which is zero.

Still another type is

$$\Gamma_{\alpha\alpha\alpha}(\Gamma_{\alpha\rho\alpha}\Gamma_{\alpha\sigma\alpha}\ldots)\Gamma_{\nu\alpha\nu}f(\rho, \sigma, \nu \ldots) \tag{e}$$

Using the fact that $\Gamma_{\alpha\rho\alpha}$ is independent of α, we get that the expression is again proportional to $(-a^2 - b^2 - c^2 + d^2)$ and is, thus, zero.

Finally, we have

$$\Gamma_{\alpha\alpha\alpha}\Gamma_{\alpha\alpha\alpha}(\Gamma_{\alpha\rho\alpha}\Gamma_{\alpha\beta\alpha}\ldots)f(\beta, \rho \ldots) \tag{f}$$

This gives a factor $-a^2 - b^2 - c^2 + d^2$ and again in zero.

The expression $\Gamma_{\alpha\alpha\alpha}\Gamma_{\sigma\alpha\sigma}\Gamma_{\rho\alpha\rho}\Gamma_{\nu\alpha\nu}$ cannot occur. Here, α appears in the middle position more than twice. That this term does not appear can be seen explicitly as follows: If we form

$$\Gamma_{\alpha\beta\gamma}\Gamma_{\gamma\sigma\beta}\Gamma_{\rho\alpha\rho} = \Gamma_{\alpha\alpha\alpha}\Gamma_{\alpha\sigma\alpha}\Gamma_{\rho\alpha\rho}$$

we get $\Gamma_{\alpha\alpha\alpha}$ appearing, since γ is the same as β (see (2.4)). We can also get $\Gamma_{\alpha\alpha\alpha}$ another way. That is, from

$$\Gamma_{\alpha\beta\beta}\Gamma_{\rho\alpha\rho}\ldots = \Gamma_{\alpha\alpha\alpha}\Gamma_{\rho\alpha\rho}\ldots$$

In the first of these expressions after we have set $\alpha = \gamma$ in the first gamma and $\gamma = \beta$ in the second gamma, there is one free (unpaired) α, which can be placed in a middle position (the $\Gamma_{\rho\alpha\rho}$ term). Thus, we end up with two α's in middle positions. In the second situation, after we have set $\alpha = \beta$ in the first gamma, we have one free α index which can be placed in a middle position as is done above. Here, too, we end up with two α's in middle positions. There is no way to get more than two α's in middle positions.

We can get chains of $\Gamma_{\alpha\nu\alpha}$, as follows:

$$\Gamma_{\alpha.\nu}\Gamma_{\nu.\rho}\Gamma_{\rho.\sigma}\ldots = \Gamma_{\alpha.\alpha}\Gamma_{\alpha.\alpha}\Gamma_{\alpha.\alpha}\ldots$$

We still have only two free indices α and σ. They could be matched up with an α and σ appearing in middle positions. Thus, here too, we see that α appears in middle positions but twice.

We have constructed all expressions that involve α appearing in the middle positions twice. We cannot have α appearing in a middle position but once, and still have an even number of α's appearing in the entire expression (since the first and third indices are the same and thus, for example, $\Gamma_{\alpha.\rho}\Gamma_{.\alpha.} = \Gamma_{\alpha.\alpha}\Gamma_{.\alpha.}$, which has an odd number of α's).

Thus, the only expressions possible involving α, have α appearing in two middle positions.† We have written down all of these and they are all zero

† Note, since some index must always occur in a middle position, and since we are free to call this index α, the results we have are general.

since they are proportional to $-a^2 - b^2 - c^2 + d^2$. Thus, we conclude all invariants involving Γ^i_{jk} are zero.†

We point out that this example, although nontrivial, is rather degenerate. After we make the transformation (2.1), we find the following structure for Γ^i_{jk},

$$\begin{aligned}
\Gamma^1_{11} &= \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^0_{10} = a' \\
\Gamma^1_{21} &= \Gamma^2_{22} = \Gamma^3_{23} = \Gamma^0_{20} = b' \\
\Gamma^1_{31} &= \Gamma^2_{32} = \Gamma^3_{33} = \Gamma^0_{30} = c' \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \Gamma^0_{00} = d'
\end{aligned} \tag{2.9}$$

with a', b', c', d' different from a, b, c, d. All the remaining Γ^i_{jk} are zero. On use of the field equations, we get the same structure maintained. Thus, a', b', c', d' changes from point to point. Henceforth, for convenience, we drop the prime in (2.9). The field equations for a, b, c, d are

$$\begin{array}{llll}
\dfrac{\partial a}{\partial x} = a^2 & \dfrac{\partial b}{\partial x} = ab & \dfrac{\partial c}{\partial x} = ac & \dfrac{\partial d}{\partial x} = ad \\[2ex]
\dfrac{\partial a}{\partial y} = ba & \dfrac{\partial b}{\partial y} = b^2 & \dfrac{\partial c}{\partial y} = bc & \dfrac{\partial d}{\partial y} = bd \\[2ex]
\dfrac{\partial a}{\partial z} = ca & \dfrac{\partial b}{\partial z} = bc & \dfrac{\partial c}{\partial z} = c^2 & \dfrac{\partial d}{\partial z} = cd \\[2ex]
\dfrac{\partial a}{\partial x^0} = da & \dfrac{\partial b}{\partial x^0} = db & \dfrac{\partial c}{\partial x^0} = cd & \dfrac{\partial d}{\partial x^0} = d^2
\end{array} \tag{2.10}$$

The solution of these equations is

$$a = -\frac{k_1}{D} \quad b = -\frac{k_2}{D} \quad c = -\frac{k_3}{D} \quad d = -\frac{k_0}{D} \tag{2.11}$$

where

$$D = xk_1 + yk_2 + zk_3 + x^0 k_0 + k_5 \tag{2.12}$$

and

$$k_1{}^2 + k_2{}^2 + k_3{}^2 - k_0{}^2 = 0 \tag{2.13}$$

This system is not finite at all points since there are points for which D vanishes.

The reason we have such degeneracy in this example is that $\Gamma^\alpha_{\beta\gamma}$ has the following structure

$$\Gamma^\alpha_{\beta\gamma} = \delta_\gamma{}^\alpha \phi_\beta \tag{2.14}$$

† If the arguments appear involved, the reader can check it out by writing down a product of very many gammas. Then, he can assign indices in a random manner such that all indices are paired up. Then, make use of the fact that the first index is the same as the third. Then, it will be clear that only the combinations listed above will appear.

with

$$\phi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{2.15}$$

and

$$a^2 + b^2 + c^2 - d^2 = 0 \tag{2.16}$$

Using (2.14), we can see right away that all invariants constructed from Γ^i_{jk} must be zero. We have introduced a rather long proof of this effect, since this lengthier discussion is useful in our future examples where $\Gamma^\alpha_{\beta\gamma}$ does not have degenerate structure (2.14).

3. *Second Example*

We can get a nondegenerate theory (all Γ^i_{jk} nonzero) by working in a higher dimensional space. For example, let us consider an 8 dimensional theory. We take

$$\begin{aligned}
\Gamma^1_{11} &= \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = a \\
\Gamma^1_{21} &= \Gamma^2_{22} = \Gamma^3_{23} = \Gamma^4_{24} = b \\
\Gamma^1_{31} &= \Gamma^2_{32} = \Gamma^3_{33} = \Gamma^4_{34} = c \\
\Gamma^1_{41} &= \Gamma^2_{42} = \Gamma^3_{43} = \Gamma^4_{44} = d \\
\Gamma^5_{55} &= \Gamma^6_{56} = \Gamma^7_{57} = \Gamma^8_{58} = a' \\
\Gamma^5_{65} &= \Gamma^6_{66} = \Gamma^7_{67} = \Gamma^8_{68} = b' \\
\Gamma^5_{75} &= \Gamma^6_{76} = \Gamma^7_{77} = \Gamma^8_{78} = c' \\
\Gamma^5_{85} &= \Gamma^6_{86} = \Gamma^7_{87} = \Gamma^8_{88} = d'
\end{aligned} \tag{3.1}$$

with

$$\begin{aligned}
a^2 + b^2 + c^2 - d^2 &= 0 \\
a'^2 + b'^2 + c'^2 - d'^2 &= 0
\end{aligned} \tag{3.2}$$

We can check that the forms (a), (b), (c), (d), (e), (f) are still zero. Thus, again all invariants formed from Γ^i_{jk}, and using g_{ij} as a metric, are zero. Also, we can check that the integrability equations are still satisfied. Using eight 8-dimensional $e^\alpha{}_i$, it is a simple matter to obtain from (2.1) a set of Γ^i_{jk} that are all nonzero. The Γ^i_{jk} vary from point to point using the field equations. It is not clear that the solutions are bounded, but at least we have eliminated the degeneracy problem.

This example is also unrealistic in that it involves eight dimensions.

4. *Third Example*

We get another nondegenerate theory when $\Gamma^\alpha_{\beta\gamma}$ have the following nonzero values

$$\begin{aligned}
\Gamma^3_{33} &= \Gamma^0_{30} = a \\
\Gamma^0_{00} &= \Gamma^3_{03} = b
\end{aligned} \tag{4.1}$$

with

$$a^2 - b^2 = 0 \tag{4.2}$$

This set of data has a two-dimensional substructure. (It is still a four-dimensional theory since $e^\alpha{}_i$ are four-dimensional). As this $\Gamma^\alpha_{\beta\gamma}$ has the same structure that appears in the proof in Section 2 and since the quantities (a), (b), (c), (d), (e), (f) are all zero, we get that all the invariants involving $\Gamma^\alpha_{\beta\gamma}$ are zero. The invariants are preserved by $e^\alpha{}_i$ transformations. The difficulty with this set of data is that we have not been able to find a maximum for g_{00}. For all the varied sets of $e^\alpha{}_i$ we have tried, we have obtained (Muraskin, 1971b) $\det A_{ab} = 0$ (within computer accuracy). We then went through the following steps hoping to avoid this difficulty. We preceded 300 points down the x-axis. At this point, we subjected the resulting Γ^i_{jk}, g_{ij} to a second $e^\alpha{}_i$ transformation. We then proceeded 300 points down the y-axis. Again we subjected the fields to a different $e^\alpha{}_i$ transformation. The procedure was repeated for the z and x^0 axis. This procedure brings in four sets of $e^\alpha{}_i$ parameters and it gets us away from looking for a maximum directly with the simple system (4.1). The integrability conditions are still satisfied at the end of the procedure since the field equation and the $e^\alpha{}_i$ transformation preserve the integrability equations. All the invariants also remain zero in this process. However, after all this was done, we still found that the resulting $\det A_{ab}$ is zero. Thus, our computer results indicate that the data is not consistent with a maximum in g_{00}.

5. *Fourth Example*

Another nondegenerate set of data results from $\Gamma^\alpha_{\beta\gamma}$ having the following nonzero values

$$\begin{aligned} \Gamma^2_{22} &= \Gamma^3_{23} = \Gamma^0_{20} = a \\ \Gamma^3_{33} &= \Gamma^0_{30} = \Gamma^2_{32} = b \\ \Gamma^0_{00} &= \Gamma^3_{03} = \Gamma^2_{02} = c \end{aligned} \tag{5.1}$$

with

$$a^2 + b^2 - c^2 = 0 \tag{5.2}$$

Again, all invariants involving $\Gamma^\alpha_{\beta\gamma}$ are zero. The data has a basic three-dimensional substructure. We found no change from Section 4 so far as a maximum in g_{00} is concerned.

All the examples proposed are nontrivial.

6. *Discussion*

We have been studying for some time the field theory in which all tensors, all orders of derivatives, all scalars are treated in a uniform manner. We may term this field theory an 'aesthetic' type field theory. In this paper, we have studied a simple set of boundary conditions. Actually, the boundary conditions we would really prefer would be $\Gamma^i_{jk} \to 0$ at spatial and temporal infinity. It is necessary for all invariants involving gamma to be zero at the origin for these boundary conditions to be satisfied. It is not clear though, that the vanishing of the invariants is sufficient to imply the desired boundary conditions.† In view of the fact that we have obtained several examples for the vanishing of invariants, it may not be too much to expect that there exists some set of data at the origin for which $\Gamma^i_{jk} \to 0$ at spatial and temporal infinity.

We may take note at this point that the aesthetic field theory has already survived some impressive hurdles.

(1) The integrability equations represent many more equations than unknowns. We have shown, nevertheless, that nontrivial solutions to these equations exist.

(2) The integrability equations arise from the requirement that mixed derivatives of any tensor function be symmetric. Now, this could lead to a different condition for each kind of tensor. This would amount to an infinite number of integrability equations which would not be favorable for a nontrivial solution. On the contrary, we have found that one does not get an ever-increasing number of restrictions.

(3) The covariant derivatives of all tensor functions vanishing would also lead to an infinite number of restrictions. Again, this is not the case.

(4) The boundary conditions require that an infinite number of invariants involving Γ^i_{jk} all be zero. We have, in fact, shown in this paper that there are at least several ways that this condition may be met.

These are some encouraging signs that the boundary condition problem may be solvable.

We may ask, does not the vanishing of the gammas at spatial and temporal infinity imply an infinite universe? The answer to this is yes and no. The universe is certainly infinite in that the fields are nonvanishing throughout finite space and time. However, this does not preclude a bounded system involving particles. That is, if one goes far enough away from a particle, one may, after a finite displacement, arrive at a point for which there are no more particles no matter how much farther one goes. The fields could fall off rapidly outside the particle system. From an experimental point of view, we would say that the universe is finite.

It is difficult to understand from a logical point of view how the universe could come out of nothing at some time zero. In our theory, such a situation would not occur as we may contemplate a 'particle universe' emerging from a vacuum characterized by nonvanishing fields. Thus, we do not get something out of nothing.

Although a geometric field theory can be mapped onto a flat space locally, this is not the case globally. Thus, the boundary conditions would be different in the curved space theory as contrasted with our theory. The question would thus be, which theory is fundamentally more reasonable so

† In equations (2.12) and (2.11) we do have $\Gamma^i_{jk} \to 0$ at spatial and temporal infinity.

far as boundary conditions are concerned? Presumably the attractive feature of the curved space theory is the possibility of a closed universe. But, if the closure appears in the time coordinate, it would appear that there would be paths that can return us back to the present instant in time. This may be looked upon as a disturbing feature. If the closure is in space only, and not in time, then the curved space theory which is supposedly introduced for the sake of bringing in boundary conditions would have nothing to offer us so far as understanding boundary conditions in time. Thus, we conclude that the aesthetic field theory has some definite advantages over the curved space theory, from a conceptual point of view.

In our work, the universe is infinite in extent from a mathematical point of view even though we anticipate a finite 'particle universe'. One may ask, should not the concept of infinity be avoided in a basic theory? We point out, on the contrary, that the concept infinity already appears in various acceptable ways. For example, we have convergent series that have an infinite number of terms in them. There are an infinite number of points in any finite interval. Thus, the notion of infinity is not *per se* a liability. We note, in curved space theory, one has a mathematical singularity associated with the derivative of the field normal to the curved surface within the framework of an imbedding space. We do not see how some kind of mathematical (as contrasted to empirical) singularity can be avoided in any case.

7. *Conclusion*

The requirement that all invariants involving gammas are zero everywhere (where the field is finite) amounts to an infinite number of restrictions. Despite this, we have found several examples of data at the origin such that these conditions are met. The vanishing of all invariants is necessary if the natural boundary conditions $\Gamma^i_{jk} \to 0$ at spatial and temporal infinity are to be satisfied. The examples we have found have not been shown to be free of difficulties. The kinds of difficulties we have encountered in our examples are: the inability to obtain a maximum (minimum) in g_{00}; the presence of singularities; and the apparent unphysical nature of higher dimensions.

Acknowledgement

The author wishes to thank B. Ring for doing the computer programming.

References

Muraskin, M. (1970). *Annals of Physics*, **59**, 27.
Muraskin, M. (1971a). *Journal of Mathematical Physics*, **12**, 28.
Muraskin, M. (1971b). *International Journal of Theoretical Physics*, Vol. 4, No. **1**, p. 49.
Muraskin, M. and Clark, T. (1970). *Annals of Physics*, **59**, 19.
Muraskin, M. and Ring, Beatrice (1971). Preprint.
Muraskin, M. and Ring, Beatrice (1972). *International Journal of Theoretical Physics* (preceding paper).
Muraskin, M. (1972). *Foundations of Physics*, **2**, 181.